A Parameter Dependent Spacecraft Guidance Boundary Value Problem

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Approximate expressions for the trajectory initial conditions as functions of the parameters are obtained for the class of boundary value problems that occur in the targeting of trajectories affected by uncertain dynamical parameters. To compute the coefficients in these expressions, sensitivity methods are employed. To reduce the computational burden, the sensitivity equations are initialized at the terminal point of the trajectory. These results are applied to a Mars lander mission. For a specific case, an accuracy analysis of the second-order approximations is outlined. Numerical integration techniques for the entry dynamics and associated variational equations are compared.

Nomenclature

A = reference surface area
$A(x,\boldsymbol{\eta})$ = sensitivity system matrix
$a,b,c,$ = atmospheric density parameters
$B(x_T,x_E)$ = first-order updating matrix
$C(x_T,x_E)$ = second-order updating matrix
C_D = drag coefficient
g = local gravitational acceleration
h = reference altitude
$\mathbf{h}_j(x,\boldsymbol{\eta})$ = first-order sensitivity forcing vector
m = mass of entry capsule
N = scale factor
$\mathbf{p}_{jk}^{(i)}(x,\boldsymbol{\eta},s)$ = class i second-order sensitivity vector
$\mathbf{q}_{jk}^{(i)}(x,\boldsymbol{\eta},s)$ = class i second-order sensitivity forcing vector
R_0 = radius of planet
s = present value of independent variable
T_0 = surface temperature of Mars planetary atmosphere
t = time
$\mathbf{u}_j(x,\boldsymbol{\eta},s)$ = first-order parameter sensitivity vector
V = velocity
v = dimensionless velocity
$\mathbf{w}_j(x,\boldsymbol{\eta},s)$ = first-order state variable sensitivity vector
x = independent variable (normalized altitude in the Mars application)
y = altitude
$\mathbf{z}$ = state vector(θ,v,Ω in the Mars application)
$\boldsymbol{\alpha}_k,\boldsymbol{\beta}_k,\boldsymbol{\gamma}_k$ = temporary storage vectors
Γ = adiabatic lapse rate of planetary atmosphere
γ = specific heat ratio of planetary atmosphere
$\boldsymbol{\eta}$ = parametric vector
θ = flight path angle
ρ = local density
σ_1 = h/N
σ_2 = $(g_0R_0)^{1/2}$
σ_3 = g_0R_0
σ_4 = $\frac{1}{2}(C_DA/m)g_0R_0$
Φ = state transition matrix
Ω = range angle

Subscripts

E = initial point of trajectory
T = final point of trajectory

Received July 6, 1970; presented as Paper 70-1016 at the AIAA Guidance, Control, and Flight Mechanics Conference, Santa Barbara, Calif., August 1970; revision received May 3, 1971. Work supported in part at Rensselaer Polytechnic Institute under NASA Grant NGR-33-018-191. The authors wish to express gratitude to R. A. Nichols, TRW Systems Group, and R. A. Surveyor, Rensselaer Polytechnic Institute for generating the various numerical results.

Index categories: Navigation, Control, and Guidance Theory; Computer Technology and Computer Simulation Techniques.

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Introduction

GUIDANCE boundary value problems have a fundamental role in the targeting of spacecraft trajectories. The determination of the partials of the velocity-required,[1,2] in which a reference initial point, final point, and connecting trajectory are given, is particularly well known. Typically, the velocity required is a function of the initial position and time and defines a ballistic trajectory from the initial position to the specified final position.

In this paper, a new parameter dependent guidance boundary value problem is considered. The initial position is fixed but some parameter affecting the spacecraft acceleration is varied. Such a parameter is a constant appearing in an atmospheric density model or in a gravitational potential function. Given the reference values of the parameters, the reference trajectory, and some perturbation in the parameter, the problem is to find the velocity-required to achieve the desired terminal condition. In view of the complex missions such as Viking and Grand Tour[3] that are currently being planned to planets surrounded by uncertain atmospheres and gravitational fields, guidance problems of this type will become increasingly important.

By employing the concept of sensitivity analysis,[4–6] it is possible to obtain closed-form approximate expressions for the trajectory initial conditions in terms of the known parameter deviations and in terms of the first- and second-order sensitivity matrices which are determined by numerical methods. The closed-form nature of these solutions and the convenience with which the solutions may be updated are advantageous since simplicity of on-board implementation will be an important consideration.

The problem of minimizing the computational effort required to generate the sensitivity matrices is investigated.

Table 1 Order of system to be solved

Accuracy of solution	Direction of integration	
	Forwards	Backwards
First order [Eq. (5)]	$n(n + m + 1)$	$n(m + 1)$
Second order [Eqs. (5) and (18)]	$\frac{1}{2}n(n + m + 1) \times (n + m + 2)$	$\frac{1}{2}n(m + 1) \times (m + 2)$

Initial conditions are imposed on the sensitivity equations at the terminal point instead of at the initial point of the reference trajectory. This significantly reduces the number of sensitivity equations that must be solved for both first- and second-order solutions of the parameter dependent guidance boundary value problem.

The problem is compared with the velocity-required problem considered by Battin[1] and Culbertson.[2] Specifically, we compare the formulations which result when the sensitivity equations are integrated from the terminal point backwards. For the velocity-required problem with a realistic gravity model, a matrix Ricatti equation must be solved if the goal is to minimize the number of differential equations to be integrated. For the present problem, it is shown that only the original parameter sensitivity equations, a linear system, must be integrated backwards. More important is the difference in the nature of the initial conditions to be applied to the sensitivity equations. For the velocity-required problem, the initial conditions do not exist. Though it is possible to avoid this difficulty in the first-order case by integrating the inverse of the desired matrix,[1] this until recently remained a problem if higher order solutions were desired.[7] For the problem considered in this paper, the initial conditions take a more convenient form, thus avoiding the aforementioned difficulties.

In addition, a detailed example involving planetary atmospheric entry is considered. For the specific case, the general results given in Table 1 are used to evaluate the relative advantage, as measured by the order of the system that must be integrated, of solving in the "forward" and "backward" directions. In addition, the accuracy of the second-order approximations is evaluated for two different terminal velocities. Finally, the advantages of two different numerical integration algorithms are compared for the atmospheric re-entry problem.

First-Order Partials of Initial Conditions with Respect to Parameters

In this section a first-order approximation for those changes in the state vector initial conditions that will zero out the terminal state vector deviations caused by the off-nominal parameters is given. The problem can be stated as follows. The state vector

$$\mathbf{z}(x) = \mathbf{g}[x, x_E, \mathbf{z}(x_E), \mathbf{n}] \tag{1}$$

is assumed to be the solution of the differential equation

$$dz/dx = \mathbf{f}(x, \mathbf{z}, \mathbf{n}) \tag{2}$$

where the symbols were defined previously (see Nomenclature). Given some perturbation in the parametric vector $\Delta\mathbf{n}$, the problem is to determine that $\Delta\mathbf{z}_E$ such that

$$\mathbf{g}[x_T, x_E, \mathbf{z}(x_E) + \Delta\mathbf{z}_E, \mathbf{n} + \Delta\mathbf{n}] = \mathbf{g}[x_T, x_E, \mathbf{z}(x_E), \mathbf{n}] \tag{3}$$

where $[x_E, \mathbf{z}(x_E)]$ is assumed to denote a specific set of initial conditions, $\mathbf{n}$ is assumed to be a specific value of the parametric vector, and x_T is assumed to be constant. To obtain a first-order solution to this boundary value problem, the first-order partial derivative matrix $\partial\mathbf{z}_E/\partial\mathbf{n}$ is calculated by differentiating Eq. (3) with respect to the jth component of $\mathbf{n}$. The result is

$$\partial\mathbf{z}_T/\partial\eta_j + [\partial\mathbf{z}_T/\partial\mathbf{z}_E]\partial\mathbf{z}_E/\partial\eta_j = \mathbf{0} \tag{4}$$

which can be solved to give

$$\partial\mathbf{z}_E/\partial\eta_j = -[\partial\mathbf{z}_T/\partial\mathbf{z}_E]^{-1}\partial\mathbf{z}_T/\partial\eta_j \tag{5}$$

To evaluate the right-hand side of Eq. (5), solutions for both the state transition matrix $[\partial\mathbf{z}_T/\partial\mathbf{z}_E]$ and the parameter sensitivity vector $(\partial\mathbf{z}_T/\partial\eta_j)$ are required. These partial derivatives are obtained by solving the following fundamental differential equations[8]

$$(d/dx)[\partial\mathbf{z}/\partial\mathbf{z}_E] = J(x)[\partial\mathbf{z}/\partial\mathbf{z}_E] \tag{6}$$

and

$$(d/dx)[\partial\mathbf{z}/\partial\eta_j] = J(x)[\partial\mathbf{z}/\partial\eta_j] + (\partial\mathbf{f}/\partial\eta_j) \tag{7}$$

where $J(x)$ is the Jacobian matrix $[\partial\mathbf{f}/\partial\mathbf{z}]$ of $f(x,\mathbf{z},\mathbf{n})$ with respect to $\mathbf{z}$ at $\mathbf{z} = g[x, x_E, \mathbf{z}(x_E), \mathbf{n}]$. Initial conditions for Eq. (6) and (7) are given by

$$[\partial\mathbf{z}/\partial\mathbf{z}_E]|_{x=x_E} = I \ [I = \text{Identity Matrix}] \tag{8}$$

and

$$[\partial\mathbf{z}/\partial\eta_j]|_{x=x_E} = 0 \tag{9}$$

Note also that the determinant of $[\partial\mathbf{z}/\partial\mathbf{z}_E]$ is not equal to zero[8] which allows the inverse of $[\partial\mathbf{z}_T/\partial\mathbf{z}_E]$ to be used. If $\mathbf{z}$ and $\mathbf{n}$ are of dimension n and m, respectively, then integrating the system of Eqs. (1, 6, and 7) forward from x_E to x_T involves $n + n^2 + nm$ differential equations. In addition, a matrix inversion and a matrix multiplication are required to complete the computation of the partial derivative matrix $[\partial\mathbf{z}_E/\partial\mathbf{n}]$.

However, it is also possible to compute the partial derivatives $\partial\mathbf{z}_E/\partial\eta_j$ by integrating Eq. (7) backwards from x_T to x_E subject to the initial condition

$$[\partial\mathbf{z}/\partial\eta_j]|_{x=x_T} = \mathbf{0} \tag{10}$$

This approach to the computation of the partial derivative matrix $[\partial\mathbf{z}_E/\partial\mathbf{n}]$ requires the simultaneous numerical integration of Eq. (1) and Eq. (7) subject to the initial condition given in Eq. (10). Thus only $n + nm$ differential equations must be integrated.

That the elements of $[\partial\mathbf{z}_E/\partial\mathbf{n}]$ may be computed by integrating Eq. (7) backwards follows directly from the definition of the sensitivity coefficients.[8] However, this result can also be demonstrated in a concise manner. To initiate the development, Eq. (5) is differentiated to give

$$\frac{d}{ds}\left(\frac{\partial\mathbf{z}_s}{\partial\eta_j}\right) = -\frac{d}{ds}\left[\frac{\partial\mathbf{z}_T}{\partial\mathbf{z}_s}\right]^{-1}\frac{\partial\mathbf{z}_T}{\partial\eta_j} - \left[\frac{\partial\mathbf{z}_T}{\partial\mathbf{z}_s}\right]^{-1}\frac{d}{ds}\left(\frac{\partial\mathbf{z}_T}{\partial\eta_j}\right) \tag{11}$$

Noting that $[\partial\mathbf{z}_T/\partial\mathbf{z}_s]^{-1} = [\partial\mathbf{z}_s/\partial\mathbf{z}_T]$ so that $[\partial\mathbf{z}_T/\partial\mathbf{z}_s]^{-1}$ is still a solution of Eq. (6) (with a different initial condition) and that (see Refs. 5 and 8)

$$(d/ds)(\partial\mathbf{z}_T/\partial\eta_j) = -[\partial\mathbf{z}_T/\partial\mathbf{z}_s](\partial\mathbf{f}/\partial\eta_j) \tag{12}$$

Equation (11) can be rewritten as

$$(d/ds)(\partial\mathbf{z}_s/\partial\eta_j) = J(s)(\partial\mathbf{z}_s/\partial\eta_j) + \partial\mathbf{f}/\partial\eta_j \tag{13}$$

which has exactly the same form as Eq. (7).

To summarize, integrating in the backward direction reduces the total number of equations to be integrated from $n^2 + n + mn$ to $nm + n$ for the first-order partials for the problem with n states and m parameters.

Second-Order Partial of Initial Conditions with Respect to Parameters

Since problems involving significant gravitational or atmospheric effects are generally nonlinear, the authors obtained closed form solutions for the second-order partials of the initial conditions with respect to the parameters. These partials are used in a second-order approximation for the changes in the state vector initial conditions that will null out terminal state vector deviations due to off-nominal parameters. The form of this approximation is

$$\Delta \mathbf{z}(x_E) = B(x_T,x_E)\Delta\mathbf{n} + C(x_T,x_E)\Delta^2\mathbf{n} \quad (14)$$

where the quantity $C(x_T,x_E)$ is the matrix of the second-order partial derivatives, $\Delta^2\mathbf{n}$ is the vector of the second-order parameter deviations, and the matrix $B(x_T,x_E)$ is the matrix of first-order partials $[\partial\mathbf{z}_E/\partial\mathbf{n}]$ that was derived in the previous section. This section discusses the computation of the C matrix.

To obtain the second partial derivatives of the initial condition vector $\mathbf{z}_E$, we differentiate Eq. (4) with respect to η_k again using the chain-rule. The following results

$$\begin{aligned}\mathbf{0} = \mathbf{p}_{jk}^{(1)} + [\mathbf{p}_{1j}^{(2)},\ldots,\mathbf{p}_{nj}^{(2)}](\partial\mathbf{z}_E/\partial\eta_k) + \\ [\partial\mathbf{z}_T/\partial\mathbf{z}_E](\partial^2\mathbf{z}_E/\partial\eta_j\partial\eta_k) + \{[\mathbf{p}_{1k}^{(2)},\ldots,\mathbf{p}^{(2)}_{nk}] + \\ ([\mathbf{p}_{11}^{(3)},\ldots,\mathbf{p}_{1n}^{(3)}]\partial\mathbf{z}_E/\partial\eta_k,\ldots,[\mathbf{p}_{n1}^{(3)},\ldots, \\ \mathbf{p}_{nn}^{(3)}]\partial\mathbf{z}_E/\partial\eta_k)\}\partial\mathbf{z}_E/\partial\eta_j \quad (15)\end{aligned}$$

where the $\mathbf{p}_{jk}^{(i)}$ are the various second derivatives of $\mathbf{z}_T$ according to the definitions

$$\mathbf{p}_{jk}^{(1)} = \frac{\partial^2}{\partial\eta_j\partial\eta_k}\mathbf{z}_T,\ \mathbf{p}_{jk}^{(2)} = \frac{\partial}{\partial\eta_k}\mathbf{w}_j,\ \mathbf{p}_{jk}^{(3)} = \frac{\partial}{\partial z_{Ek}}\mathbf{w}_j \quad (16)$$

where $\mathbf{w}_j$ is the jth column of $[\partial\mathbf{z}_T/\partial\mathbf{z}_E]$ and z_{Ek} is the kth element of $\mathbf{z}_E$. See Ref. 6 for the differential equations of the second-order sensitivity coefficients and further discussion. To simplify the following results, we define the auxiliary variable $\boldsymbol{\alpha}_r$ as

$$\boldsymbol{\alpha}_r = \mathbf{p}_{rk}^{(3)} + [\mathbf{p}_{r1}^{(3)},\ldots,\mathbf{p}_{rn}^{(3)}](\partial\mathbf{z}_E/\partial\eta_k) \quad (17)$$

Substituting Eq. (17) into Eq. (15) and solving the result for $\partial^2\mathbf{z}_E/\partial\eta_j\partial\eta_k$ gives

$$\begin{aligned}\partial^2\mathbf{z}_E/\partial\eta_j\partial\eta_k = -[\partial\mathbf{z}_T/\partial\mathbf{z}_E]^{-1}\{\mathbf{p}_{jk}^{(1)} + \\ [\mathbf{p}_{1j}^{(2)},\ldots,\mathbf{p}_{nj}^{(2)}](\partial\mathbf{z}_E/\partial\eta_k) + \\ [\boldsymbol{\alpha}_1,\ldots,\boldsymbol{\alpha}_n](\partial\mathbf{z}_E/\partial\eta_j)\} \quad (18)\end{aligned}$$

Note that Eq. (18) gives the second derivatives of the initial conditions for a fixed terminal condition and that both Eqs. (5) and (18) are required since Eq. (5) gives $\partial\mathbf{z}_E/\partial\eta_k$ and $\partial\mathbf{z}_E/\partial\eta_j$.

Computing $\partial^2\mathbf{z}_E/\partial\eta_j\partial\eta_k$ by evaluating the right-hand side of Eq. (18) requires numerical integration of the differential equations for the second-order sensitivity coefficients. For the case with n states and m parameters, there are a total of $\frac{1}{2}n\ (n + m)(n + m + 1)$ second-order sensitivity coefficients. Solving Eqs. (1, 6, and 7), and the differential equations for the $\mathbf{p}_{jk}^{(i)}$ (forward integration from x_E to x_T) thus requires the numerical integration of a system of $\frac{1}{2}n\ (n + m + 1)(n + m + 2)$ differential equations. By comparison, direct evaluation of $\partial^2\mathbf{z}_E/\partial\eta_j\partial\eta_k$ by integrating the second-order sensitivity equations in the backwards direction requires the computation of only $\frac{1}{2}n(m)\ (m + 1)$ second-order coefficients. In this approach, a total of $\frac{1}{2}n\ (m + 1)\ (m + 2)$ numerical integrations are required for a second-order solution to the parameter dependent guidance boundary value problem. Table 1 summarizes the results with respect to the relative advantages of integrating in the backwards direction.

Application to Mars Viking Mission[9,10]

Consider a Mars mission consisting of a transfer trajectory, insertion into a parking orbit around Mars, de-orbit, atmospheric entry, and landing. Repeated performance of the radio occultation experiment[11] during successive orbits could update the Martian density parameters prior to de-orbit and atmospheric entry. Given the updated atmospheric parameters, the problem is to find the perturbed entry conditions required to achieve the reference terminal conditions. The second-order solution to this problem is

$$\begin{bmatrix}\Delta\theta_E\\ \Delta v_E\\ \Delta\Omega_E\end{bmatrix} = B(x_T,x_E)\begin{bmatrix}\Delta a\\ \Delta b\\ \Delta c\end{bmatrix} + C(x_T,x_E)\begin{bmatrix}\frac{1}{2}(\Delta a)^2\\ \Delta a\Delta b\\ \Delta a\Delta c\\ \frac{1}{2}(\Delta b)^2\\ \Delta b\Delta c\\ \frac{1}{2}(\Delta c)^2\end{bmatrix} \quad (19)$$

where θ = flight-path angle, v = normalized velocity, Ω = range angle, x_E = normalized entry altitude, x_T = normalized terminal altitude, and $\Delta a,\Delta b,\Delta c$ = update minus reference atmospheric parameters. The columns of the B and C matrices are the solutions of the first- and second-order sensitivity equations for Mars entry. See the Appendix for the equations of motion for atmospheric re-entry. To evaluate the relative advantage of integrating in the backwards direction, the expressions of Table 1 are used with n (the number of states) = 3 and m (the number of parameter) = 3. For the second-order expression given in Eq. (19), solution in the forward direction requires a total 84 elements to be integrated while solution in the backward direction requires only 30 elements to be integrated.

Because problems involving atmospheric re-entry are usually quite nonlinear, it is reasonable to inquire as to the accuracy of the expansion given in Eq. (19). This accuracy study can be performed in the following manner. Given a reference trajectory from which the B and C matrices can be computed and a set of perturbations Δa, Δb, Δc, the vector of corrections $(\Delta\theta_E,\Delta v_E,\Delta\Omega_E)$ can be computed. This corrected trajectory could be flown to the terminal altitude and the resulting terminal states compared against the reference values to compute an error index. Repetition of this procedure for different sets of perturbations would give the relationship between the parameter perturbations and the terminal error after the correction has been applied. With knowledge of the expected variations in the atmospheric density parameters, the expected variations in the terminal states resulting from the "corrected" trajectory can be evaluated.

With this in mind, the perturbation region around the nominal value of the parametric vector $\mathbf{n} = [a,b,c]$ for which the expansion of Eq. (19) gives acceptable results, was investigated. For the entry vehicle configuration and reference trajectory simulated, the size of the perturbation region depends on the reduction in velocity achieved between the initial and final points. This is illustrated in Table 2. In particular, the performance of the second-order solution [Eq. (19)] was superior with higher terminal velocities. This reflects the nonlinearity of the low-velocity flight regime.[12] Physically, the perturbation Δa = 0.9 in Table 2

Table 2 Size of perturbation region vs velocity at terminal point[a]

Perturbation	Terminal velocity $V_T = 0.523\ V_E$	Terminal velocity $V_T = 0.150\ V_E$
Acceptable positive Δa perturbation[b]	Δa,[c] = +0.9	Δa = +0.5
Acceptable negative Δa perturbation	Δa = −0.5	Δa = −0.2

[a] Reference trajectory based on the following data: $\hat{V}_E$ = 12,300 fps; $\hat{y}_E$ = 89,400 ft; $\hat{\theta}_E$ = 0.15287 rad; $\hat{\Omega}_E$ = 0; m/C_DA = 0.30 slugs/ft²; $\hat{a}$ = −10.2348; $\hat{b}$ = 2.7027; $\hat{c}$ = −0.501556.
[b] For a perturbation less than the given limits, retargeting according to Eq. (19) will reduce the terminal errors.
[c] The quantity $\Delta a = \ln[\rho_0(\text{actual})/\rho_0(\text{Ref.})]$ where ρ_0 is the surface density.

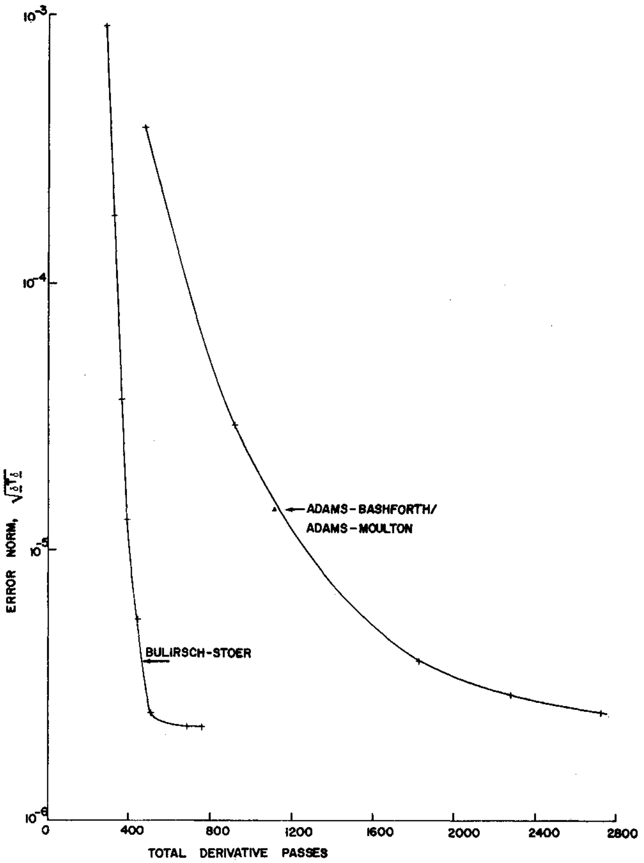

Fig. 1 Error norm vs total derivative passes for a fixed trajectory segment.

corresponds to an actual surface density almost twice as large as the reference value.

Computational Techniques and Results

In the previous section, Eqs. (5) and (18) were used to reduce the number of sensitivity coefficients that are necessary to compute the elements of the B and C matrices in Eq. (19). (see Table 1.) However, the problem of numerically integrating a system of 30 first-order differential equations is not trivial. For this reason, the relative performance of various numerical integration packages in computing the solution of this system of 30 differential equations is studied.

For the Mars Entry application, the system has the form

$$(d/dx)\mathbf{X} = \mathbf{F}(x,\mathbf{X}) \tag{20}$$

where $\mathbf{X}$ is a vector with 30 elements. The components of $\mathbf{X}$ are, respectively, $\mathbf{z}(x)$ which satisfies Eq. (A4) and the various first- and second-order coefficients. Numerical values for the vehicle and the atmospheric parameters required to compute $\mathbf{F}$ in Eq. (20) are tabulated in Table 2. Finally, the following initial conditions are assumed for x at $x_T = 0.67213$:

$$X_1 = 0.811609, \quad X_2 = 0.0572567, \quad X_3 = 0.02852$$
$$X_j = 0., \qquad j = 4,\ldots, 30 \tag{21}$$

and $\mathbf{X}$ is computed at $x_E = -0.46557$. Note that this trajectory does include the "aerodynamically-dominated"[12] flight regime.

To compare the relative performance of several subroutines in integrating Eq. (20) subject to the initial conditions given in Eq. (21), it is necessary to define two additional quantities[13]: a measure of the error in the computed state vector $X(x_E)$ and a measure of the cost of integrating the trajectory with a particular technique and error control to obtain $\mathbf{X}(x_E)$. The measure of the error is defined to be the norm of the absolute error by

$$\text{Error Norm} = (\boldsymbol{\delta}^T\boldsymbol{\delta})^{1/2} \tag{22}$$

where the absolute error vector $\boldsymbol{\delta}$ is given by

$$\boldsymbol{\delta} = \mathbf{X}(x_E) - \mathbf{X}^*(x_E) \tag{23}$$

and the quantity $\mathbf{X}^*(x_E)$ should be the true solution. Note that this error measure is at least the size of the maximum individual error among the thirty states. The cost of integrating the trajectory to obtain $\mathbf{X}(x_E)$ is defined to be the total number of derivative passes required.

For the atmospheric re-entry problem which is defined by Eqs. (20) and (21) and does not possess an analytic solution, the true solution $\mathbf{X}^*(x_E)$ is approximated by using the standard Runge-Kutta-Gill technique with a step-size of $1.25(10)^{-4}$ which required a total of 36408 derivative passes.‡ Note that now the quantity $\mathbf{X}(x_E) - \mathbf{X}^*(x_E)$ is a good estimate of the absolute error only when it is much larger than the difference between $\mathbf{X}^*(x_E)$ and the true solution.

Performances of the Bulirsch-Stoer extrapolation technique[14] and more common techniques such as the Runge-Kutta-Gill and the Adams-Bashforth Adams-Moulton[15] are compared in Table 3. These preliminary results indicate that the Bulirsch-Stoer algorithm requires a small fraction of the derivative passes required by RKG or Adams-Moulton for equivalent accuracy.

Finally, in Fig. 1, the error norm $(\boldsymbol{\delta}^T\boldsymbol{\delta})^{1/2}$ is plotted vs the total number of derivative passes. The asymptotic effect noted for small values of the error norm suggests that the difference between $\mathbf{X}^*(x_E)$ and the true solution is not insignificant when the error norm is small.

Conclusions

A new parameter dependent guidance boundary value is considered in which a parameter affecting the spacecraft acceleration is not well known. Since missions are currently being planned to planets surrounded by uncertain atmospheres and gravitational fields, guidance problems of this type will become increasingly important.

Sensitivity analysis is employed to give closed-form approximate solutions for the trajectory initial conditions as functions of the parameter deviations. The closed-form nature

Table 3 Numerical integration comparisons

Technique	Stepsize/error tolerance	$(\boldsymbol{\delta}^T\boldsymbol{\delta})^{1/2}$	Derivative passes
Runge-Kutta-Gill (4th order)	Stepsize $h = 0.01$	$4.6(10)^{-4}$	456
	$h = 0.005$	$2.7(10)^{-5}$	912
	$h = 0.0025$	$5.2(10)^{-7}$	1820
Adams-Bashforth/ Adams-Moulton (4th order)	Stepsize $h = 0.005$	$3.8(10)^{-4}$	464
	$h = 0.0025$	$2.9(10)^{-5}$	918
	$h = 0.00125$	$3.9(10)^{-6}$	1828
	$h = 0.001$	$2.9(10)^{-6}$	2284
	$h = 0.000833\ldots$	$2.5(10)^{-6}$	2738
Bulirsch-Stoer	Tolerance:		
	TØL = 10^{-3}	$9.2(10)^{-4}$	276
	TØL = 10^{-4}	$1.8(10)^{-4}$	316
	TØL = $.5(10)^{-4}$	$3.7(10)^{-5}$	356
	TØL = 10^{-5}	$1.3(10)^{-5}$	380
	TØL = 10^{-6}	$5.5(10)^{-6}$	436
	TØL = 10^{-7}	$2.5(10)^{-6}$	500
	TØL = 10^{-8}	$2.2(10)^{-6}$	685
	TØL = 10^{-9}	$2.2(10)^{-6}$	749

‡ All computations reported in this section were done on a CDC 6400 digital computer with a 60 bit word length in single precision arithmetic.

of these solutions is advantageous since simplicity of onboard implementation will be an important consideration.

The problem of obtaining the modified sensitivity coefficients is investigated. Initial conditions are imposed on the sensitivity equations at the terminal point instead of at the initial point of the reference trajectory. This significantly reduces the number of sensitivity equations that must be solved for both first- and second-order solutions of the parameter dependent guidance boundary value problem.

Applications of this analysis to a Mars probe mission was investigated. Also, the computational aspects were considered. Significant reduction in the computer time required to integrate the planetary entry dynamics and associated sensitivity equations is achieved through application of an advanced numerical integration technique.

Appendix: Mars Entry Model

For two dimensional, unpowered, ballistic flight without lift, assuming a spherically symmetric planet surrounded by a nonrotating atmosphere, the time domain equations of motion can be written as follows[16]:

$$dy/dt = -V\sin\theta$$

$$d\theta/dt = -(1/V)[g(y) - V^2/(R_0 + y)]\cos\theta$$

$$dV/dt = -\tfrac{1}{2}\rho(y)V^2 C_DA/m + g(y)\sin\theta$$

$$d\Omega/dt = V\cos\theta/(R_0 + y) \tag{A1}$$

where both the gravity and drag terms have been included. To reduce the order of the system, a normalized altitude is taken to be the independent variable:

$$x = [(h - y)/h]N \tag{A2}$$

Also, the actual velocity is divided by the circular satellite velocity resulting in the dimensionless variable.

$$v = V/(g_0R_0)^{1/2} \tag{A3}$$

With these assumptions, the entry dynamics [Eq. (A1)] can be rewritten

$$d\theta/dx = (\sigma_1/\sigma_2 v\sin\theta)\{[g(x)/\sigma_2 v - v\sigma_2/[R_0 + h(1 - x/N)]\cos\theta\}$$

$$dv/dx = (\sigma_1/\sigma_3 v\sin\theta)\{-\sigma_4\rho(x)v^2 + g(x)\sin\theta\}$$

$$d\Omega/dx = \sigma_1\cos\theta/R_0 + h(1 - x/N) \tag{A4}$$

Because most aerobraking occurs at low altitudes in Mars entry, the following adiabatic model[17] is assumed for the atmospheric density

$$\rho(x) = e^a[1 + c(1 - x/N)]^b \tag{A5}$$

where the atmospheric density parameters a, b, c are

$$a = \ln\rho_0,\quad b = 1/(\gamma - 1),\quad c = \Gamma h/T_0 \tag{A6}$$

and x was defined in Eq. (A2).

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